# A Study of Fuzzy Ideals and Fuzzy Rings

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**(SYNOPSIS)**

The Thesis entitled " A Study of Fuzzy Ideals and Fuzzy Rings" deals with the Fuzzification of the results in the theory of classical rings and ideals.

The notion of a fuzzy subset was introduced by L.A Zadeh [23].His seminal paper in 1965 has opened up new insights and applications in a wide range of scientific fields. Azriel Rosenfeld [17] used the notion of a fuzzy subset to set down cornerstone papers in several areas of Mathematics, among other disciplines. Rosenfeld [17] applied this concept of a Fuzzy subset to the theory of groupoids and groups. Kuroki [8] has studied fuzzy ideals and bi-ideals in semi groups. Wan-Jin-Liu [21] has studied Fuzzy ideals of a ring. It was followed by the studies of Mukherjee and Sen [14], who defined and examined fuzzy prime ideals of a ring. Fuzzy prime ideals were further investigated by Malik and Mordeson [12], and they gave a complete characterization of fuzzy prime ideals of an arbitrary ring. Fuzzy ideals over Artinian rings have been studied by Malik [9], where he gave a characterization of Artinian rings in terms of Fuzzy ideals.

This Thesis consists of five chapters. Some definitions, notations and preliminary results from classical rings and ideals and also some basic definitions and results of Fuzzy rings and Fuzzy ideals, which are required in the sequel, are included in chapter 1.

The characteristic function of a crisp set assigns a value of either 1 or 0 to each individual in the universal set, thereby discriminating between members and non-members of the crisp set under consideration. This function can be generalized such that the values assigned to the elements of the universal set fall within a special range and indicate the membership grade of these elements in the set. Larger values denote higher degrees of set membership. Such a function is called a membership function and the set defined by it a fuzzy set. A *Fuzzy set* $\mu$ of X is defined as a function $\mu : X \to [0,1]$. If A and B are any two fuzzy subsets of X, then we can define $A \subseteq B$ if and only if $A(x) \le B(x)$, $\forall x \in X$ and $A = B$ if and only if $A(x) = B(x)$, $\forall x \in X$.

If $(R,+,\cdot)$ is a ring and I is a subset of R, then $(I,+,\cdot)$ is an ideal of the ring

$(R,+,\cdot)$ if and only if (i). $a,b \in I \Rightarrow a-b \in I$ (ii). $a \in I$ and $r \in R \Rightarrow a\,r \in I$ and $r\,a \in I$. Also we have, if I is an ideal of a ring R, then I is a subring of R. But the converse of this result is not true. For example, in the ring of rationals $(Q,+,\cdot)$, the system $(Z,+,\cdot)$ is a subring of integers of the ring $(Q,+,\cdot)$. But the ring of integers $(Z,+,\cdot)$ is not an ideal. We have the union and intersection of any two ideals is again an ideal.

A fuzzy subset $\mu$ of a ring R is called a *fuzzy ring* on R, if it satisfies, $\forall x,y \in R$

(i). $\mu(x-y) \geq \min\{(\mu(x), \mu(y)\}$ and (ii). $\mu(x.y) \geq \min\{(\mu(x), \mu(y)\}$

A fuzzy subset $\mu$ of a ring R is called a fuzzy left ideal if we have the following :

(i). $\mu(x-y) \geq \min\{(\mu(x), \mu(y)\}$ and (ii). $\mu(x.y) \geq \mu(y)$, $\forall x,y \in R$.

If we replace axiom (ii) by $\mu(x.y) \geq \mu(x)$, $\forall x,y \in R$, then $\mu$ becomes a *fuzzy right ideal.*

If $\mu$ is a fuzzy left and a fuzzy right ideal of R, then $\mu$ is called a *fuzzy ideal* of R. We have the intersection of a family of fuzzy left (right) ideals of R is again a fuzzy left(right) ideal of R.

In chapter II, we discuss fuzzy prime ideals and fuzzy semi prime ideals. A non-constant fuzzy ideal P of a ring R is said to be *prime*, if for any two fuzzy ideals A and B of R, $A \circ B \subseteq P \Rightarrow$ either $A \subseteq P$ or $B \subseteq P$. It is observed that I is a prime ideal of R, if and only if the characteristic function $\chi_I$ is a fuzzy prime ideal on R. Further, if P is a fuzzy prime ideal of a ring R and if $P_0 = \{x/x \in R$ and $P(x) = P(0)\}$, then $P_0$ is a prime ideal in R. Then there exist an integer $n \geq 0$ such that $P_0 = nZ$, where Z is the ring of integers. Also, P can take atmost r values where r is the number of distinct positive divisors of n. The following theorem gives a complete characterization of fuzzy prime ideals on Z. If P is a non – null fuzzy prime ideal on Z, then P has two distinct values. Conversely, if P be a fuzzy subset of Z such that $P(n) = \alpha_1$, when p / n and $P(n) = \alpha_2$, when p n, then P is a fuzzy prime ideal on Z. As an application to Ring theory, we have, if R is a ring with identity 1 and if every fuzzy ideal on R has finite values then R is a Noetherian Ring. A Fuzzy ideal $\mu$ of a commutative ring R is called *a fuzzy semi prime ideal* if $\mu(x^2) = \mu(x)$, $\forall\ x \in R$. We have, if $\mu$ is a fuzzy ideal of a commutative ring R, then $\mu$ is a fuzzy semi prime ideal of R, if and only if its level ideals $\mu_t = \{x \in R / \mu(x) \geq t\}$ are semi prime ideals for all $t \in Im\ \mu$. Further an ideal I ($\neq$Z) of Z is semi prime if and only if I = {0} or I = nZ, for some square free integer n.

In Chapter III, we present fuzzy maximal ideals of a ring R and observe these properties. A non-constant fuzzy ideal A is a *fuzzy maximal ideal* of R, if for any fuzzy ideal B of R, $A \subseteq B \Rightarrow$ either $A_* = B_*$ or $B = \chi_R$ . It is proved that if A is a fuzzy maximal ideal of R, then A(0) = 1. Also, if A is a fuzzy maximal ideal of R, then $|Im\ (A)| = 2$. Further if A is a fuzzy maximal ideal of R, then $A_*$ is a maximal ideal of R. We observed that if A is a fuzzy ideal of R and $A_*$ is a maximal ideal of R, then A is two –valued. Also as a characterization of fuzzy maximal ideals, we have if A is a fuzzy ideal of R and if $A_*$ is a maximal ideal of R such that A(0) = 1, then A is a fuzzy maximal ideal of R.

In Chapter IV, we characterize fuzzy prime ideals and fuzzy maximal ideals in the special case, where R is the ring of rational numbers Q. If I is an ideal of the ring of rational numbers Q and if $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$, then the fuzzy subset A defined by,

$$A(x) = \begin{cases} \beta \; ; \text{ if } x \in I \\ \alpha \; ; \text{ other wise,} \end{cases}$$ is a fuzzy ideal of Q.

As a characterization of fuzzy prime ideals on the ring of rational numbers Q, we prove the following theorem :

Let I be a prime ideal of the ring of rational numbers Q and $\alpha \in [0,1]$ and let P be the fuzzy subset of Q defined by,

$$P(x) = \begin{cases} 1 \; ; \text{ if } x \in I \\ \alpha \; ; \text{ otherwise }. \end{cases}$$

Then P is a fuzzy prime ideal of Q.

Further we obtained the result : if P be a fuzzy subset of Q, then P is a fuzzy prime ideal of Q if and only if, there exist a prime ideal I of Q and an element $\alpha \in [0,1)$ such that

$$P(x) = \begin{cases} 1 \; ; \text{ if } x \in I \\ \alpha \; ; \text{ otherwise }. \end{cases}$$

Thus we obtained a theorem, which determines all fuzzy prime ideals of the ring of rational numbers Q. That is, the fuzzy prime ideals of the ring of rational numbers Q are just the fuzzy ideals P given by,

$$P(q) = \begin{cases} 1 \; ; \; \text{if} \; p/q \\ \alpha \; ; \; \text{otherwise,} \end{cases}$$

where $p$ is a prime integer or 0 and $q$ is a rational number in Q and $\alpha < 1$.

By a maximal fuzzy ideal of the ring of rational numbers Q, we mean, as usual, a maximal element in the set of all non constant fuzzy ideals of Q. Then we determine all maximal fuzzy ideals of Q, by establishing a one-to-one correspondence between the maximal fuzzy ideals of Q and the pairs (M, $\alpha$) where M is the maximal ideal of Q and $\alpha$ is an element in [0,1]. Further we prove that a fuzzy subset A of the ring of rational numbers Q is a maximal fuzzy ideal of Q if and only if there exist a maximal ideal M of Q and an element $\alpha \in [0,1)$ such that,

$$A(x) = \begin{cases} 1 \; ; \; \text{if} \; x \in M \\ 0 \; ; \; \text{otherwise} \; . \end{cases}$$

In Chapter V, we introduce Q-fuzzy rings and Q – fuzzy ideals of a ring and derive their fundamental properties and then proceed to obtain their characterizations. If R is a ring and Q is the ring of rational numbers, then R is called a *Q-ring* if,

i. $q.a \in R \quad \forall a \in R$ and $q \in Q$

ii. $q(a+b) = qa + qb \quad \forall a,b, \in R$ and $q \in Q$

iii. $q(a\,b) = (qa)b = a\,(q\,b)$

The ring of complex numbers C, the ring of real numbers R. the ring of rational numbers Q etc. are all examples of a Q-ring.

But the ring of integers Z, $nZ = \{0 \pm n, \pm 2n, \pm 3n, \ldots\ldots\ldots\}$ and the ring $Zp = \{0,1,2,\ldots\ldots\ldots p-1\}$ are all examples of rings which are not Q-rings.

An ideal I in a Q- ring R, is said to be a *Q- ideal* if $q.a \in I \quad \forall q \in Q$ and $a \in I$.

A proper Q-ideal P of a Q-ring R is called a *Q-Prime ideal* if for all pairs of Q-ideals A and B of R, $A \circ B \subseteq P \Rightarrow$ either $A \subseteq P$ or $B \subseteq P$. If R is a Q-ring and A is a fuzzy ring on R such that $A(qx) \geq A(x) \quad \forall x \in R$ and $q \in Q$, then A is said to be a *Q-fuzzy ring* on R. Then we proved that if A and B are any two Q-fuzzy rings on R, then $A \cap B$ is also a Q-fuzzy ring on R. Also we showed that A is Q-fuzzy ring of R if and only if $A_t$ is a Q – subring of R for any $t \in [0,1]$. Then we defined Q-fuzzy ideal and Q-fuzzy prime ideal and obtained a characterization of Q-fuzzy prime ideals on a Q-ring R as follows :

Let I be a Q-ideal of a Q-ring R, $\alpha \in [0,1)$ and $\mu$ be a fuzzy subset on R defined by,

$$\mu(x) = \begin{cases} 1 \; ; \text{ if } x \in I \\ \alpha \; ; \text{ if } x \notin I \end{cases}$$

Then $\mu$ is a Q-fuzzy prime ideal of R if and only if I is a Q-prime ideal of R.

As a corollary, we have, if I is a Q-ideal of a Q-ring R, then the characteristic function $\chi_I$ of I is a Q-fuzzy prime ideal of R, if I is a Q-prime ideal of R.

Then we observed the following properties of the Q-fuzzy ideal $\mu$ of R.

i. $\mu(0_R) = 1$

ii. $\text{Im}\,\mu = \{1, \alpha\}$ where $\alpha \in [0,1)$.

iii. $\mu_0 = \{ x \in R \,/\, \mu(x) = \mu(0_R) \}$ is a Q – Prime ideal of R.

In this final Chapter we also discuss Q-fuzzy homomorphisms, quotient structures and ascending (descending) chain conditions in the context of Q-fuzzy rings.