1 Introduction

Image processing refers to the analysis and manipulation of a digitized image. Image segmentation is a major part of image processing that attempts to partition an image into distinct regions containing each pixels with similar attributes so that the analysis and interpretation of the images could be made easier. The research on image segmentation has been of great attention for many years. Thousands of segmentation techniques have been developed in the recent past. Different features of an image are extracted through segmentation and analyzed more closely. The level to which the extraction is done depends on the complexity of the problem being solved. Most of the algorithms for image segmentation are based on the two basic properties of image intensity values - discontinuity and similarity. Even though several segmentation algorithms of nontrivial images have already been done by researchers, it still remains an area worth investigating in the realm of image processing. Traditional segmentation techniques focus on partitioning imagery into image-objects with well-defined boundaries.

In this thesis, fuzzy mathematical tools are used in the analysis of image segmentation.Measures of fuzziness and image information can be used in segmentation and thresholding tasks. If we interpret the image features as linguistic variables, then we can use fuzzy if-then rules to segment the image into different regions. Fuzzy image segmentation techniques are advantageous over classical methods as they are capable of handling imprecise and vague data in which uncertainty is intrinsically present. Fuzzy set approach gives a gradual transition between the overlapping boundaries. Fuzzy IF-THEN rules are being increasingly applied in image segmentation. The advantages of the fuzzy rules based image segmentation over other methods are mainly that humans can more easily understand the problems due to linguistic representation of numeric variables, it is computationally less expensive than fuzzy clustering methods[2], and it has the potential ability to integrate the domain of expert knowledge.

A fuzzy subset in a set X, as proposed by Lotfi A Zadeh in 1965, is as a function $A : X \rightarrow [0, 1]$, $A(x)\, for\, x\ \in X$ represents the degree of the membership of x in A. The study of fuzzy image processing as the name suggests, is the method of applying different fuzzy techniques in image processing. A general fuzzy image processing scheme is given in [1].

Edge detection is a common problem in image processing. This problem has been broadly treated and documented since the early stages of image segmentation. The main challenge on edge detection is to determine which pixel belongs to the edge. Uncertainty related to the edge composition is very high in the neighbourhood of the edge. Based on the similarity of intensity values, the images are partitioned into similar regions. Since the images are captured from different sources of energy, one algorithm cannot solve all segmentation problems. In this work, we also attempt to develop some methods which are also be applicable to the set of images.

2 Summary of the thesis

The thesis is organized into seven chapters. The opening chapter, Chapter 0 is the Introduction, in which a brief outline of the thesis is given.

Chapter 1 contains all the basic concepts and results required for the rest of the thesis.

Chapter 2 deals with fuzzy divergence measures which is used to quantify the difference between two fuzzy sets. In this chapter, we define a fuzzy divergence measure based on the different cardinalities of the fuzzy set.We observe that, out of the different cardinalities of fuzzy sets, the scalar and fuzzy cardinalities could be used to find the dissimilarities among two fuzzy sets and we have the following definitions:

Definition: For two fuzzy sets $A, B \in \mathbf{F}(\mathbf{X})$, the amount of information for discrimination in favour of A against B is defined as

$$F(A, B; \alpha_i) = \frac{1}{|A| + |B|} max \left\{ 0, \left| \left| \widetilde{A} \right|_{\alpha_i} - \left| \widetilde{B} \right|_{\alpha_i} \right| \right\}$$

Definition: For two fuzzy sets $A, B \in \mathbf{F}(\mathbf{X})$, the fuzzy expected information for discrimination in favour of A against B is given as

$$F_1(A, B) = \frac{1}{|A| + |B|} \sum_i \alpha_i max \left\{ 0, \left| \left| \widetilde{A} \right|_{\alpha_i} - \left| \widetilde{B} \right|_{\alpha_i} \right| \right\}$$

Definition: For two fuzzy sets $A, B \in \mathbf{F}(\mathbf{X})$, the amount of information for discrimination of A' against B' is given as

$$F_2(A', B') = \frac{1}{|A| + |B|} \sum_i (1 - \alpha_i) max \left\{ 0, \left| \left| \widetilde{A} \right|_{1-\alpha_i} - \left| \widetilde{B} \right|_{1-\alpha_i} \right| \right\}$$

Definition: The total amount of fuzzy information for discrimination in favour of A against B is defined by

$$F(A,B) = F_1(A,B) + F_2(A,B)$$

$$= \frac{1}{|A|+|B|}\sum_i \alpha_i max\left\{0, \left|\left|\widetilde{A}\right|_{\alpha_i} - \left|\widetilde{B}\right|_{\alpha_i}\right|\right\}$$

$$+\frac{1}{|A|+|B|}\sum_i (1-\alpha_i) max\left\{0, \left|\left|\widetilde{A}\right|_{1-\alpha_i} - \left|\widetilde{B}\right|_{1-\alpha_i}\right|\right\}$$

$$= \frac{1}{|A|+|B|}\sum_i \left[\alpha_i\, max\left\{0, \left|\left|\widetilde{A}\right|_{\alpha_i} - \left|\widetilde{B}\right|_{\alpha_i}\right|\right\}\right] + (1-\alpha_i)\, max\left\{0, \left|\left|\widetilde{A}\right|_{1-\alpha_i} - \left|\widetilde{B}\right|_{1-\alpha_i}\right|\right\}$$

Definition: For two fuzzy sets $A, B \in \mathbf{F(X)}$, Fuzzy divergence between A and B is given by

$$D_R(A,B) = F(A,B) + F(B,A)$$

$$= \frac{1}{|A|+|B|}\sum_i \left[\alpha_i max\left\{0, \left|\left|\widetilde{A}\right|_{\alpha_i} - \left|\widetilde{B}\right|_{\alpha_i}\right|\right\} + \alpha_i max\left\{0, \left|\left|\widetilde{B}\right|_{\alpha_i} - \left|\widetilde{A}\right|_{\alpha_i}\right|\right\}\right]$$

$$= \frac{2}{|A|+|B|}\sum_i \left[\alpha_i max\left\{0 \left|\left|\widetilde{A}\right|_{\alpha_i} - \left|\widetilde{B}\right|_{\alpha_i}\right|\right\}\right]$$

It has been proved that the above definition is a distance measure as well as a metric on fuzzy sets.Also, this measure is related to Kosko's [5] subsethood measure and then it is reduced to Bhandari and Pal's [6] fuzzy divergence measure. Later on we observe that the defined measure satisfies all the properties of $\sigma-$ distance measure [7]. Further we define a fuzzy entropy based on this fuzzy divergence measure. The theorems and proposition we obtained are given below:

Theorem: $D_R(A,B)$ is a distance measure on $\mathbf{F(X)}$.

Theorem: $D_R(A,B)$ is a metric on $\mathbf{F(X)}$

Proposition: (Relation between Kosko[5] and defined measure)

$$S(A;B) = \frac{\sum count(A \cap B)}{K \sum countA} D_R(A;B),$$

where

$$K = \sum_i \alpha_i max \left\{ 0, \left| \left| \widetilde{A} \right|_{\alpha_i} \right| - \left| \left| \widetilde{B} \right|_{\alpha_i} \right| \right\}$$

Theorem: $D_R(A;B)$ reduces to $D(A;B)$[6]

Theorem: $D_R(A;B)$ is a $\sigma-$distance measure[7] on $\mathbf{F}(\mathbf{X})$

Definition: The entropy measures the amount of fuzziness of a fuzzy set and it can be defined as

$$H(A) = \frac{D_R(A, A_{near})}{D_R(A, A_{far})}$$

Also in this chapter, we develop a segmentation method based on the fuzzy divergence measure. The analysis of the test images shows that the method is a better segmentation method compared to other methods [3],[11].

Chapter 3 deals with the uncertainty measures based on the different types of fuzzy cardinalities and its connection to the fuzzy divergence measure given in Chapter 2. Also we attempt to develop a relation between the divergence measure given earlier and the uncertainty measure emerged through the relative fuzzy cardinality. Consequently, a new class of divergence measures is developed.

The important definitions and results are listed below:

Definition: Given a finite set X with A as its fuzzy set, then the relative cardinality of A is defined as

$$Card(A) = \frac{A}{X} = \frac{\sum \mu_A(x)}{|X|}$$

Definition: Uncertainty quantity of fuzzy set A is defined as $U(A) = -\log_2 Card(A)$

Definition: Given a finite set X and a fuzzy set, A on X we calculate the average uncertainty quantity of the fuzzy set with $H(A) = -\sum_{i=1}^{n} log_2 Card(A)$

Here we obtain the following relations between uncertainty measure and the fuzzy divergence measures.

$$U(A) = -\log_2 \left(\frac{1}{n[D_R(A;B)]} \sum_i \left[\alpha_i\, max \left\{ 0 \left| \left| \widetilde{A} \right|_{\alpha_i} - \left| \widetilde{B} \right|_{\alpha_i} \right| \right\} \right] \right) \rightarrow (1)$$

$$U(A) = - - \log_2 \left(\frac{2}{n[D_R(A;B)]} \sum_i \left[\alpha_i\, max \left\{ 0 \left| \left| \widetilde{A} \right|_{\alpha_i} - \right| \right\} \right] \right) \rightarrow (2)$$

$$U(A) = -\log_2 \left(\frac{2}{n[D_R(A;B)]} \sum_i \left[\alpha_i\, max \left\{ 0 \left| \left| \widetilde{B} \right|_{\alpha_i} - \right| \right\} \right] \right) \rightarrow (3)$$

$$U(A+B) = -log_2 \left(\frac{2}{n[D_R(A;B)]} \sum_i \left[\alpha_i\, max \left\{ 0 \left| \left| \widetilde{A} \right|_{\alpha_i} - \left| \widetilde{B} \right|_{\alpha_i} \right| \right\} \right] \right) \rightarrow (4)$$

Theorem: $U(A) = -\log_2 card(A)$ satisfies all the axioms of nonspecificity given in[4][8].

Proposition: $D_R(A;B) = \frac{1}{n}\sum_i \left[\alpha_i max\left\{0\left|\left|\widetilde{A}\right|_{\alpha_i} - \left|\widetilde{B}\right|_{\alpha_i}\right|\right\}\right]$

Proposition: $D_R(A;B) = \frac{2}{n}\sum_i \left[\alpha_i max\left\{0\left|\left|\widetilde{A}\right|_{\alpha_i}\right|\right\}\right]$

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Proposition: $U(A+B) < U(A) + U(B)\, if\, U(A) = U(B)$

Proposition: $D_R\left(A' \cup C, B' \cup C\right) = D_R\left(A \cap C', B \cap C'\right) \forall A, B, C \in (F(X))$

Proposition: For any two fuzzy subsets of X, $D_R(A \cup B, A \cap B) = D_R(A, B)$

Proposition: Let A,B,C be three fuzzy subsets of X then

$D_R(A \cup B, C) \leq D_R(A, C) + D_R(B, C)$

Further we developed an image segmentation method based on the threshold of an image. The threshold is selected by calculating the fuzzy divergence measure between two subsets of an image.

Segmentation based on fuzzy binary relation for medical images is studied in Chapter 4. Medical images mostly contain complicated structures and their precise segmentation is necessary for clinical diagnosis. A segmentation algorithm is developed and illustrated using test medical images.The proposed algorithm is given below:

Algorithm

Input: X, R

Output: S

Step 1: Input the original image,$X = \{x_1, x_2, ..., x_n\}$ *which is a subset of*R^m*,* *where n is the number of distinct pixels of X and m = 2 for gray scale images or m = 3 for colour images using RGB space.*

Step 2: Use the default relation or input user defined relation.

Step 3: Find the inverse relation γ^{-1} *of* γ

Step 4: Compute a fuzzy similarity relation Q (Step 2 and Step 3)

Step 5: Choose an $\alpha \in \Lambda(Q)$

Step 6: Obtain an equivalence relation α_Q *from Step 5*

Step 7: Take the image clusters as equivalence classes obtain under the partition induced by α_Q*.*

Step 8: Generate the segmented image S with respect to image clusters, obtained in step 7 and thus stops algorithm.

In Chapter 5 we consider near sets introduced by J.F.Peters[9][10] and the idea of near fuzzy sets is introduced using fuzzy similarity relation discussed earlier. A probe function is defined on the set of perceptual objects as equivalence classes.

Some of definitions and results obtained are summarized below:

Definition: Let O be the set of perceptual objects and $\phi : O \to R$ is given by $\phi_x = \frac{|[x]|}{|\phi/[x]|}$, where $|[x]|$ is the cardinality of each equivalence classes and $|\phi/[x]|$ denote the number of equivalence classes in each partition.

Definition: A fuzzy perceptual system $\langle O, \mathbf{F} \rangle$ consists of a nonempty set **O** of sample perceptual objects and a nonempty set F of probe functions.

Definition: Let $\langle O, \mathbf{F} \rangle$ be a fuzzy perceptual system. $\forall B \subseteq F$, the indiscernibility relation is defined as follows:

$$\sim_B = \left\{ ([x]_{\alpha_i}, [y]_{\alpha_j}) \in O \times O \,/\, \forall O/\alpha_i \, and \, O/\alpha_j \; such \; that \; \phi_x = \phi_y \right\}$$

Definition: Let $\langle O, \mathbf{F} \rangle$ be a fuzzy perceptual system. $\forall B \subseteq F$, the weak indiscernibility relation is defined as follows:

$$\simeq_B = \left\{ ([x]_{\alpha_i}, [y]_{\alpha_j}) \in O \times O \,/\, \exists \, O/\alpha_i \, and \, O/\alpha_j \; such \; that \; \phi_x = \phi_y \right\}$$

Definition: Let $\langle O, \mathbf{F} \rangle$ be a fuzzy perceptual system and $X, Y \subseteq O$. A set X is weakly near fuzzy (ϖ) to set Y within the fuzzy perceptual system

$$\langle O \, , \, F \rangle \Leftrightarrow \exists \, an \, x \in X \, and \, y \in Y \, such \, that \, x \simeq_B y \, for \, some \, B \subseteq F.$$

Definition: Let O be the set of perceptual objects and $\phi : O \to R$ is given by $\phi_x = \frac{|[x]|}{|\phi/[x]|}$, where $|[x]|$ is the cardinality of each equivalence classes and $|\phi/[x]|$ denote the number of equivalence classes in each partition.

Theorem: Let $\langle O, \mathbf{F} \rangle$ be a fuzzy perceptual system and $X, Y \subseteq O.$ Then the following condition holds.

1.$Reflexivity.i.e. \forall X \neq \phi, X \varpi X$

2.$Symmetry \Rightarrow x \varpi \, y \, then \, y \varpi x$

3.$Transitivity \Rightarrow let X, Y, Z \subseteq O \, and \, X \varpi Y, Y \varpi Z \, then \, X \varpi Z$

Proposition: Let $\langle O, \mathbf{F} \rangle$ be a fuzzy perceptual system.Then quotient set,$O/\alpha_i \subseteq \aleph\Im$.

Proposition: The following properties are also true for weak near fuzzy sets.

1.$X \varpi Y \Rightarrow X \neq \phi, Y \neq \phi$

2.$X \subseteq Y \Rightarrow X \varpi Y$

A conclusion of the whole thesis and a brief account of potential problems of further investigation are given in the final chapter