2. Review of Literature

I will utilize journal-papers, books and theses, some of which are reviewed below.

1. Heijenoort, Jean Van [editor], (1967), *From Frege to Gödel: A Source Book on Mathematical Logic 1879-1931*, is a seminal and authoritative primary source, making available forty six original papers, all relating to the foundations of mathematics, including translations of the originals in German, French, etc., some made by Prof. Heijenoort himself. Particularly relevant to this research is the path breaking work of Kurt Gödel, considered as the greatest logician since Aristotle, published in 1931, originally in German, entitled “On Formally Undecidable Propositions of Principia Mathematica and Related Systems I”. This paper is about the two Incompleteness Theorems and are at the heart of all subsequent work. Since the original paper is in German, the editor has done a splendid job in translating the paper into English with the approval of Gödel himself. On p 595, the editor Heijenoort states, “The translation of the paper is by the editor, and it is printed here with the kind permission of Professor Gödel and Springer Verlag. Professor Gödel approved the translation, which in many places was accommodated to to his wishes”. This book will be a base reference for the present Thesis, as all subsequent Incompleteness Theorems are built upon and extensions of Gödel’s Theorem.

2. Feferman, Solomon.(editor-in-chief) et. al.,(1986) [Vol. I],( 1990) [Vol. II], (1995) [Vol. III], (2003) [Vol. IV], (2003) [Vol. V], have done a tremendous service to all concerned people by creating this landmark resource that puts together the lifetime work of the foremost logician of the 20th century, Kurt Gödel (1906 – 1978), into five volumes. Volumes I and II are devoted to Gödel’s publications in full, both in the original and as well as their translations into English. Volume III features a wide selection of unpublished articles and lectures. Volumes IV and V bring to a conclusion the Oxford edition of the ‘Collected Works’ of Kurt Gödel and contain some of the most significant of Gödel’s correspondences with leading logicians and mathematicians as well as letters to and from the editors and other figures.

All the five volumes contain an excellent editorial apparatus such as introductory notes that provide extensive explanatory and historical commentary on each body of work,
3. Turing, Alan M. (1936), “On computable numbers, with an application to the Entscheidungs Problem”. This Paper has to do with the “Entscheidungsproblem”, which is German for “The Decision Problem”, first proposed by David Hilbert in 1900 called the “10th Problem”. Turing proved in the above paper that the “Decision Problem” has no solution! In Turing’s own words, “In particular, it is shown (§11) that the Hilbertian Entscheidungsproblem can have no solution” (p 231). Again, on p.259, he states, “…I propose, therefore to show that there can be no general process for determining whether a given formula $\mathfrak{A}$ of the functional calculus $\mathbf{K}$ is provable, i.e. there can be no machine which, supplied with any one of $\mathfrak{A}$ of these formulae, will eventually say whether $\mathfrak{A}$ is provable”. This again, has devastating effects not only for theoretical computer science, but for the whole of mathematics. Once again this thesis will investigate the precise impact of this Theorem on mathematics.

4. Petzold, Charles, (2008), The Annotated Turing, presents a commentary on the seminal 36-page paper of Alan M. Turing, entitled “On computable numbers with an application to the Entscheidungsproblem”. Turing’s paper is written in the usual technical style and so most people, especially those untrained in mathematics and computer science, will find it hard to follow. Moreover, Turing’s paper also contains his own terminology such as a-machines, Universal Computing Machine, O-machine, etc., which are now known as Turing Machines, Universal Turing Machines, Oracle Machines, etc. respectively. Additionally, at the time of Turing’s 1936 paper, when computers had not yet been invented, a “computer” was primarily a human who “computed”. For instance, on pp. 136-137 of his paper, Turing writes, “The behaviour of the computer at any moment is determined by the symbols which he is observing, and his “state of mind” at that moment ... we may now construct a machine to do the work of this computer. ...” All of this can be very confusing to any untrained reader of Turing’s paper. Accordingly, a commentary is very necessary to clarify all these notions, as well as the technical details of the associated mathematical and computer science concepts. Charles Petzold does a marvellous job of providing a line-by-line exegetical commentary with more than 8:1 ratio of annotations of Turing’s paper. Charles Petzold’s book, The Annotated Turing is
unique, in the sense that it is the only book that explains in detail the fine points line-by-line and very often phrase-by-phrase of everything in Alan Turing’s paper. Therefore, this book is virtually indispensable to anyone, layman or professional, who wishes to reference Turing’s paper.

5. Parisi, Luciana. (2013), in *Contagious Architecture: Computation, Aesthetics, and Space* is about the most recent developments in theoretical computing and its philosophy. The book is divided into thee parts: (1) Incomputable Objects in the Age of the Algorithm, (2) Soft Extension: Topological Control and Mereotopological Space Events, and (3) Architectures of Thought. The central argument of the book is that algorithms do not exclusively channel data according to preset mechanism of binary synthesis involving 0s and 1s (p. x). Of primary relevance to this research is the first section wherein the author deals with ‘incomputable probabilities’ such as *Chaitin’s constant* Ω (Omega) [p.17]. The impact of *Omega* and the *Super-Omegas* on theoretical computing and formal axiomatic systems are investigated. This book is an up-to-date account of the state of computing and mathematics.

6. Tarski, Alfred, (1983), *Logic, Semantics, Metamathematics*. This book contains seventeen papers of Tarski from 1923 to 1938. All the seventeen papers are English translations from the originals in German, French and Polish. Under the expert editing of John Corcoran in consultation with Tarski himself, we have the only complete and reliable English translation of Tarski’s papers, which is far superior to the original translation published by Oxford University Press in 1956. In this research, we will be concerned with Tarski’s main paper, “The Concept of Truth in Formalized Languages”, originally in Polish. Tarski’s *Undefinability Theorem* is at the heart of this paper, which will be the starting point of our analysis.

7. Chaitin, Gregory. J., (1977), *Algorithmic information theory*, has in this paper, developed an information-theoretic approach to metamathematics. He is one of the originators of *Algorithmic Information Theory* (AIT), which lies at the heart of his *Incompleteness-theorem*. This paper is one of his developments in which he has reformulated and tailor-made AIT along with the notion of a *Turing-machine* and theoretic-computer versions of a *formal mathematical system* in which *axioms* are finite strings, *rules of inference* are
an algorithm for enumerating theorems, etc. Since AIT is at the center of Chaitin’s Incompleteness theorem, this research will incorporate several tools herein.

8. Chaitin, Gregory. J., (1974), *Information-theoretic limitations of formal systems*, has in this paper used information-theoretic concepts along with Chaitin’s AIT to measure the difficulty of proving a given set of theorems in terms of the number of bits of axioms that are assumed, and the size of the proofs needed to deduce the theorems from the axioms. This is a further development of the Chaitin’s previous paper and is also necessary for this research.

9. Chaitin, Gregory. J., (1989), ‘Undecidability and Randomness in Pure Mathematics’. Just as Gödel and Turing have shown that Incompleteness and undecidability lie at the heart of mathematics, Chaitin, in this work shows that Randomness too, lies at the heart of mathematics. This radically alters epistemological issues regarding the nature of mathematical truth. This, too, will be analyzed in this research.

10. Chaitin, Gregory. J., (2006), *The Limits of Reason*. This paper is written by IBM mathematician and computer scientist Gregory Chaitin, describing his journey from the ideas of complexity and randomness, originally suggested by mathematician Gottfried W. Leibniz in 1686 to his own pioneering work in the development of Algorithmic Information Theory in the 1970s, culminating in his two incompleteness theorems, particularly his second incompleteness theorem involving his own creation of an irreducible, incompressible and absolutely random, but well-defined number called the Halting Probability, denoted as $\Omega$, and named in his honour as Chaitin’s Constant. Chaitin lucidly explains how $\Omega$ encodes all the infinite truths and answers to all mathematical questions via the ‘halting- problem’. He goes on to describe a number of limitations of mathematics dictated by $\Omega$ and how there will be an infinite amount of mathematics which will be forever beyond the reach of mathematical investigation. Chaitin has done an admirable job of communicating complicated truths of his work to an intelligent layperson.

11. Calude, Cristian S., *et al.* (2002), in *Computing a Glimpse of Randomness*, have done a daring and difficult task of computing the exact values of the first 64 bits of the Halting-Probability or Chaitin’s Constant $\Omega_U$ (often abbreviated to simply $\Omega$), where U is a Universal Self Delimiting Turing Machine. The proof combines Java-Programming
with mathematical proofs. The calculated value of the first 64 bits of $\Omega_0$ is $\Omega_{64}^0 = 0.00000010000011000011101000111110111011000010000$. Mathematicians Charles Bennett and Martin Gardner have stated that, “if one could calculate the first few thousands digits of $\Omega_0$, it would contain the answers to more mathematical questions than could be written down in the entire universe”.

12. Becher, Veronica; Daicz, Sergio & Chaitin, Gregory., (2001), *A Highly Random Number*, in “Combinatorics, Computability and Logic”, C.S. Calude, M. J. Dinneen & S. Sburlan (Eds.), Springer: London, pp. 55-68. Just over a decade ago, researchers working in the field of theoretical computing and the foundations of mathematics shocked the mathematical community when they discovered that the *Halting Probability* $\Omega$ encoding all the limitations of Logic, Mathematics and Computer Science does not end with it. They have proved the existence of what they describe as ‘Super Omegas’, whose *randomness* and *uncomputability* for surpasses Chaitin’s Constant $\Omega$. In this thesis, we will also explore the import of these findings on mathematics and its philosophy.

13. Becher, V. & Chaitin, G., (2002), *Another example of Higher Order Randomness*, in “Fundamenta Informaticae”. In this paper, computer scientist Veronica Becher discovers links between Chaitin’s-constant or the *Halting Probability* $\Omega$, the higher-order omegas and real computers. She shows that the probability that an infinite computation will produce only a finite amount of ‘output’ is the same as the *Halting Probability* $\Omega$. Additionally, Becher has also that $\Omega$ is equivalent to the property that, during an infinite computation, a computer will fail to produce an ‘output’, i.e., it will obtain no result from a computation, and go on to the next one, and that the computer will do this only a finite number of times.

14. Benacerraf, Paul & Putnam, Hilary (Eds.),(1983), *Philosophy of Mathematics*, (2nd ed.), Cambridge University Press. This is an anthology of mathematical philosophy and the foundations of mathematics. The collection of essays has, for its authors, some of the foremost mathematicians and mathematical philosophers. Although it would be difficult to say just what comprises the philosophy of mathematics, nevertheless, the essays cover a very broad spectrum of views such as *logicism*, *intuitionism*, *formalism*, *Hilbert’s program*, *Empiricism*, *Platonism*, *mathematical truth*, *semantics*, *models & reality*, *cantor’s set theory*, the *nature of mathematical reasoning* and many other topics, all
related to the foundations of mathematics, as it arose early in the 20th century and progressed till the time of the publication of this volume. This anthology has been hailed by all researchers in the field as a significant and comprehensive resource for any one engaged in mathematical philosophy and will continue to be so for decades to come.

15. In *Mathematical Logic and the Foundations of Mathematics*, Kneebone (1963) has presented a clear exposition of the subject. This book has been used by every teacher and researcher in the area of Logic and Foundations of mathematics. In Part I, Kneebone expounds the basic concepts of the subject. In Part II, he covers the major developments in mathematical-logic and foundations from around 1870 to 1940. Part III is on the philosophy of mathematics which is also relevant to my research. Additionally, an Appendix deals with the developments in mathematical-logic till the time the book was written.

16. Smullyan, Raymond M.,(1992), *Gödel’s Incompleteness Theorems* is not only an outstanding authority on metamathematics, but also a skilled expositor. The book guides the reader through a series of carefully formulated proofs of the central results with a high degree of generality and insight. There is also a wonderful exposition of recursive function theory, so essential for the understanding of Gödel’s theorem. The book never confuses rigor with dullness or obscurity

17. Peter Smith (2007), in his *An Introduction to Gödel’s Theorems* has done a splendid job of communicating a highly abstruse subject and making it accessible to graduate-level. He has lucidly dealt with a variety of ways to prove Gödel’s first incompleteness theorem and exploring a family of related theorems, not easily found in the standard literature. Also addressed are the Turing-theorem, Halting-problem and theoretical computer science.

18. Boolos, George S., Burgess, John P. & Burgess, Richard Jeffrey (2002), *Computability and Logic* has become a classic reference for several reasons : first, it is written by experts and contributors to the field; second, it is “student user friendly”, offering a new and simpler treatment of the representability of recursive functions, a traditional stumbling block for many people, but a necessary topic for metamathematicians. It includes topics from *Turing’s theory* to *Ramsey’s theorem* and many mere topics,
making it a necessary reference work for any researcher working in the foundations of mathematics.

19. Shanker, Stuart G. (Ed.), (1988), *Gödel’s Theorem in Focus*, as editor, has made the work of other logicians accessible to the general mathematician and philosopher. Besides the translation of Gödel’s seminal paper on his celebrated theorems, the book includes papers and articles by other logicians/mathematicians. For instance, Stephen C. Kleene’s article “The work of Kurt Gödel” (pp 48-73), John W. Dawson’s “The Reception of Gödel’s Incompleteness Theorems” (pp 74-95), as well as several other logicians, including S. G. Shanker himself makes the book highly invaluable for any researcher in the field.

20. Mendelson, Elliott, (1979), *Introduction to Mathematical Logic*. Nearly forty years after it was first published in 1964, Elliot Mendelson’s book still remains the best textbook on the principle topics of this subject. Of particular relevance is Chapter 3, wherein are presented recursive-functions, Gödel’s Incompleteness Theorems, Tarski’s Indefinability of Truth Theorem, etc. Chapter 5 contains a rigorous notion of Turing Machines and Turing’s Theorem. And finally, the 4th edition concludes with an excellent appendix on 2nd Order Logic.

21. Ebbinghaus, H. –D, Flum, J., & Thomas, W. (1994), *Mathematical Logic* in this extremely important book, in Chp II, deals with the Syntax of 1st-Order Logic, which is the framework of almost all present-day mathematics as well as the foundations of mathematics. Chp VII deals with the scope of 1st-Order Logic. Chp IX explores 2nd-Order Predicate Logic and makes explicit some of the difficulties such as Incompleteness and even the problem of how closely the truth of a formula in 2nd-Order Logic depends on what we take as true in set-theory: Different axiomatizations of set-theory lead to different semantics for 2nd-Order Logic. Chp X Limitations of the Formal Method, has to do with limitative-metatheorems including a section on on *Register-Machines* (a version of Turing Machines) and a general theorem about the Undecidability of any theory that can encode the workings of a Register-Machine.

22. Hedman, Shawn (2004), *A First Course in Logic: An introduction to model theory, proof theory, computability and complexity*. This is a clear and unifying treatment of fundamental concepts underlying theoretical computer science and the foundations of
mathematics. As the subtitle indicates, model-theory, proof-theory and computability is intimately involved in this research.

23. Piccinini, Gualtiero (2003), *Computations and Computers in the Sciences of Mind and Brain*, Doctoral Thesis submitted to the University of Pittsburgh. Some of the underlying concepts in Piccinini’s Doctoral thesis such as Recursive-functions, Gödel-numbers & Universal Turing Machine Programs, Unsolvability of the Halting-Problem (HP), etc. is also involved in my research as basic.

24. Shapiro, Stewart (1997), *Philosophy of Mathematics: Structure and Ontology*, takes up the age-old debate between Platonism of the standard realist and the anti-realist, placing it in a broad philosophical perspective. Shapiro also presents his original structuralist philosophy and axiomatic framework in detail. This book is a significant contribution to the ongoing debate in mathematical philosophy and the nature of mathematical ‘objects’.

25. Simmons, Roger A. & Gold, Bonnie, Editors, (2008), *Proof and Other Dilemmas: Mathematics and Philosophy*, In the last thirty-five years or so, philosophers of mathematics began to revive basic questions concerning the philosophy of mathematics such as : What is the nature and significance of mathematical Proof ; What is the nature of mathematical ‘objects’ and ‘mathematical knowledge’. Two new schools of philosophy of mathematics, Social Constructivism and Structuralism, was were added to the four traditional views (Formalism, Intuitionalism, Logicism & Platonism ), all of which are taken up by specialists in their fields. Of particular relevance to my research is section III, Chapters 7 – 10, pp. 131-242 ; wherein the Platonistic view of mathematics is debated.

26. Li Ming & Vitányi, Paul. , (1977), *An Introduction to Kolmogorov Complexity and its Applications*, is a classic and comprehensive resource text on the notion of Kolmogorov Complexity in particular, and the notion of randomness in general, written by two experts in the field. Chapter 1, entitled ‘Preliminaries’, deals with the basic ideas of binary-strings, computability theory, randomness, etc., along with the notations to be used in the rest of the book. Chapter 2, entitled ‘Algorithmic Complexity’, is the heart of the book, dealing with random finite and infinite sequences, incompressibility, Algorithmic Information Theory, Kolmogorov – Complexity, etc. Chapter 3, entitled ‘Algorithmic Prefix Complexity’ and Chapter 4, entitled ‘Algorithmic Probability’ deal
with the details of Kolmogorov-complexity and *Algorithmic Complexity theory*. Chapter 5 through 8 deals with the applications of Kolmogorov-Complexity.

In summary, this work by Li and Vitányi is a invaluable resource for any student or researcher who seeks to understand and use *Algorithmic Information Theory*, which is indeed the case of this thesis.