1. Introduction:

For millennia, both mathematicians and non-mathematicians alike have held the view that only mathematics provides absolute knowledge, absolute certainty, which is static, perfect, rigorous, ‘cast in stone’ and absolutely true in all possible worlds! In fact, world renowned mathematician David Hilbert had put forward a proposal in the early 1920s to secure the foundations of all of mathematics, which later came to be known as the Hilbert Program.

This program was to encapsulate all of mathematics within a formal-system $F$, which included an ‘alphabet’ (set of symbols), a ‘grammar’ (to determine which strings of symbols were ‘well-formed formulas’ or wffs), inference-rules, and, of course, a universal set of Axioms $\{A_1, A_2, A_3, \ldots A_n\}$ to be the bedrock encompassing all of mathematics, universally accepted, permanently valid & fulfilling the following requirements: 1.) Completeness of $F$ : Every True statement or proposition $\phi$ in mathematics should be provable within $F$. 2.) Consistency of $F$ : No contradictions should be derivable within $F$, & 3.) Decidability of $F$ : There should be a “Decision procedure” to decide mechanically / algorithmically, in a finite number of steps, the Truth or Falsity of any mathematical statement or wwf (well-formed formula). Most mathematicians, especially Hilbert, believed that all the above three goals were achievable.

However, in the last eighty years, beginning in 1931, a series of powerful metatheorems, known as Incompleteness Theorems, have been proved. At this point, a word about ‘Metamatematics’ in order. Almost all working, day-to-day mathematicians are involved with their area of speciality; working in a specific branch of mathematics such as Functional Analysis, Algebraic Topology, Non-associative Algebras, K-Theory …, and hundreds of other specialized areas. These mathematicians work out results within mathematics. However, metamathematicians are mathematicians who study results/metatheorems about mathematics; its semantics, syntax, foundational-logic, scope, limitations, etc. Relatively few mathematicians venture into this arena. These Incompleteness-theorems are limitative results which have serious ramifications for the foundations, superstructure and philosophy of the whole of mathematics. Logicians and mathematicians are far from fully realizing and fully working out all their consequences. These six metatheorems or incompleteness theorems are:

(1) Gödel’s First Incompleteness Theorem.
(II) Gödel's Second Incompleteness Theorem.

(III) Tarski’s Incompleteness Theorem.

(IV) Turing’s Incompleteness Theorem,

(V) Chaitin’s First incompleteness theorem, and

(VI) Chaitin’s Second incompleteness theorem

A brief account of these incompleteness theorems are outlined below:

(I) Gödel’s First Incompleteness Theorem:

Statement: If $F$ is a formal system with a decidable set of axioms such that the language of $F$ includes $\text{PA}$ (Peano-Arithmetic), then, $\exists$ a sentence $\varphi$ such that (i) If $F$ is consistent, then $F \not\vdash \varphi$ and (ii) If $F$ is $\omega$-consistent, then $F \not\vdash \sim \varphi$.

The ‘$\omega$-consistency’ condition is strictly stronger than simple consistency. It was J. B. Rosser who improvised Gödel’s Theorem and replaced $\omega$-consistency by simple consistency, thereby imparting an even greater generalization of Gödel’s Theorem. This came to be known as the Rosser- Gödel Theorem.

Gödel’s Theorem / Rosser- Gödel Theorem has an extremely deep impact with many ramifications for the whole of mathematics, i.e. the foundations, edifice (all the branches of mathematics), and even the philosophy of mathematics. Logicians/mathematicians are far from realizing all the consequences of this one theorem alone!

(II) Gödel’s Second Incompleteness Theorem:

Statement: If $F$ is a formal system with a decidable set of axioms such that the language of $F$ includes PA (Peano-Arithmetic) and $F$ is consistent, then $F \not\vdash \text{Con}_F$, where $\text{Con}_F$ is the sentence asserting the consistency of $F$.

(III) Tarski’s Incompleteness Theorem:

This theorem is also called as Tarski’s Undefinability Theorem or Tarski’s Truth Theorem.
Tarski’s Undefinability Theorem for First-order Arithmetic: If First-order Peano-Arithmetic (PA) is consistent, \( \mathcal{L}(PA) \) = language of PA & M is the standard model of PA, Then, there does not exist any truth predicate or truth definition \( T(x) \) of \( \mathcal{L}(PA) \) within PA.

Tarski’s Generalized Undefinability Theorem: If F is any consistent or sound first-order Formal-system extending Peano-Arithmetic & M is any model of F, then, the set

\[
\text{True}_M = \{ \text{GN}(\varphi)/M \models \varphi \}
\]

is not definable in M, i.e. there does not exist any truth predicate or truth definition for F.

This theorem is a semantic counterpart to Gödel’s Incompleteness Theorem. Basically, what this theorem is saying is that no sufficiently powerful formal-system is semantically self-representational.

This thesis will explore the metamathematical ramifications of this theorem for the global view of mathematics.

(IV) Turing’s Incompleteness Theorem

Statement: Given a Universal Turing Machine (UTM), a program \( p \) and an input data-set I, there is no effectively computable Decision Procedure, which will infallibly and in finite time reveal whether any given proposition \( \varphi \) is provable or not, i.e. whether the program \( p \) will halt or not, called the Halting-Problem (HP).

Theoretical computer science is inextricably and inseparably connected with mathematics; its objects of study being not just theorems but also their proofs, as well as calculations, programs, and algorithms.

We will study the ramifications of Turing’s incompleteness theorem for mathematics.

(V) Chaitin’s First Incompleteness Theorem:
This theorem is based on *Algorithmic Information Theory* (AIT), created by IBM mathematician Gregory Chaitin (1947 - ) and is intimately connected with theoretical computing, the *Halting-problem* (HP) and *Turing’s theorem*.

**Statement**: If $F$ is a *formal axiomatic system* such that (i) $F$ is *recursively axiomatizable*, (ii) $F$ contains sufficient arithmetic [*such as Peano-Arithmetic* (PA), or *Robinson-Arithmetic* (Q)], & (iii) $F$ is *consistent* and *sound*, then, there exist a constant $C_F$, such that one cannot prove any statement of the form $H(x) > C_F$ within $F$, where $H(x)$ is the *algorithmic-complexity* or *Kolmogorov-Chaitin Complexity* of $x$.

This is a significant extension of *Gödel’s First Incompleteness Theorem*. We will investigate the implications of this metatheorem for mathematics and its foundations.

**(VI) Chaitin’s Second Incompleteness Theorem**:

Chaitin had incorporated his *first incompleteness theorem* and all its ramifications in a number which he invented and named as *Omega* $\Omega$. Fascinated by Turing’s work, he began to investigate the halting problem. He considered all the possible programs that Turing’s hypothetical computer could run, and then looked for the probability that a program, chosen at random from among all the possible programs, will halt. The work took him nearly 20 years, but he eventually showed that this "halting probability" turns Turing’s question of whether a program halts into this real number $\Omega$ somewhere between 0 and 1. If $U$ is a *Universal Turing Machine* which accepts only binary prefix-free programs $p$, then

Chaitin’s constant (or the Halting Probability) $= \Omega$

$$= \sum_{p \in F} 2^{-|p|}$$

where, $F$ is the set of all programs $p$ which halt & $|p| = \text{length of the binary-program } p$.

**Statement**: If $F$ is a *formal axiomatic system* such that (i) $F$ is *recursively axiomatizable*, (ii) $F$ contains sufficient arithmetic [*such as Peano-Arithmetic* (PA), or *Robinson-Arithmetic* (Q)], & (iii) $F$ is *consistent* and *sound*, then, $F$ enables us to determine only as many bits of $\Omega$ as its *complexity*. 

Now, $\Omega$ is the most important number in all of mathematics and we will show how it encodes every mathematical *Truth*! Additionally, we will show how $\Omega$ embodies a whole cluster of limitations of mathematics. This theorem has a devastating impact on the whole of mathematics.

It takes *Turing’s Incompleteness Theorem* to dizzy heights, spawns even more limitations on mathematics and opens up a plethora of research avenues!

This thesis will spell out the details of Chaitin’s theorems and even very recent research surpassing Chaitin’s work; namely, *super-omegas*, and the associated limitations of mathematics.