2. Work Plan and Methodology

The main target of work plan is to find analytical approximate solution of fractional partial differential equations arising in several physical phenomena by using homotopy methods (such as homotopy perturbation method, homotopy analysis method, homotopy perturbation transform method and homotopy analysis transform method). The main aim of this synopsis is to check the behavior of solutions for highly nonlinear fractional differential equations arising in science, fluid flow, ecology and some other fields by using modified semi-analytical as well as numerical methods with coupling of Laplace, Elzaki and Sumudu Transforms.

a. Basic idea of homotopy perturbation method

In this method, using the homotopy technique of topology, a homotopy is constructed with an embedding parameter \( p \in [0,1] \) which is considered as a “small parameter”. This method became very popular amongst the scientists and engineers, even though it involves continuous deformation of a simple problem into a more difficult problem under consideration. Most of the
perturbation methods depend on the existence of a small perturbation parameter but many nonlinear problems have no small perturbation parameter at all. Many new methods have been proposed in the late nineties to solve such nonlinear equations devoid of such small parameters (Ganji and Rajabi (2006); He (1999); Liao (1995, 1997)). Late 1990s saw a surge in applications of homotopy theory in the scientific and engineering computations (He, (1998, 1999, 2000); Hillermeier (2001)). When the homotopy theory is coupled with perturbation theory it provides a powerful mathematical tool (He (1998, 2004); Hillermeier (2001)). A review of recently developed methods of nonlinear analysis can be found in (He (2000)). To illustrate the basic concept of HPM (Homotopy Perturbation Method), consider the following nonlinear functional equation

\[ A(u) = f(r), \quad r \in \Omega \text{ with the boundary conditions } B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \partial \Omega, \]

(1)

where \( A \) is a general functional operator, \( B \) is a boundary operator, \( f(r) \) is a known analytic function, and \( \partial \Omega \) is the boundary of the domain \( \Omega \). The operator \( A \) is decomposed as \( A = L + N \), where \( L \) is the linear and \( N \) is the nonlinear operator. Hence, (1) can be written as
\[ L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \]

We construct a homotopy \( v(r, p) : \Omega \times [0,1] \to \mathbb{R} \) satisfying

\[
H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega. \tag{2}
\]

Hence,

\[
H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{3}
\]

where \( u_0 \) is an initial approximation for the solution of (2). As

\[
H(v,0) = L(v) - L(u_0) \quad \text{and} \quad H(v,1) = A(v) - f(r), \tag{4}
\]

it shows that \( H(v, p) \) continuously traces an implicitly defined curve from a starting point \( H(u_0,0) \) to a solution \( H(v,1) \). The embedding parameter \( p \) increases monotonously from zero to one as the trivial linear part \( L(u) = 0 \) deforms continuously to the original problem \( A(u) = f(r) \). The embedding parameter \( p \in [0,1] \) can be considered as an expanding parameter (He (1999)) to obtain.

\[
v = v_0 + pv_1 + p^2v_2 + \ldots \tag{5}\]

The solution is obtained by taking the limit as \( p \) tends to 1, in Eq. (5). Hence
\[ u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \]  

(6)

The series (6) converges for most cases and the rate of convergence depends on \( A(u) - f(r) \) (He (1999)).

b. Basic idea of Homotopy analysis method

Homotopy analysis method (HAM) was first proposed by Liao (1992) based on homotopy, a fundamental concept in topology and differential geometry. The HAM is based on construction of a homotopy which continuously deforms an initial guess approximation to exact solution of the given problem. An auxiliary linear operator is chosen to construct the homotopy and an auxiliary parameter is used to control the region of convergence of the solution series, which is not possible in other methods like perturbation techniques (Cole (1968); Dyke (1975); Murdock (1991); Nayfeh (2000)), homotopy perturbation method (Dehghan et al. (2007, 2007b, 2008, 2008b, 2008c); Shakeri et al. (2008)) and other non-perturbative methods such as the artificial small parameter methods (Lyapunov (1992)), the \( \delta \)–expansion method (Karmishm et al. (1990)) and
the decomposition method (Adomian (1976, 1984, 1991, 1994); Wazwaz (2000, 2002); Ramos et al. (2001); Baboolian et al. (2002)). The HAM provides greater flexibility in choosing initial approximations and auxiliary linear operators and hence a complicated nonlinear problem can be transformed into an infinite number of simpler, linear sub-problems, as shown by Liao et al. (2007).

To describe the basic idea of HAM, we consider the following nonlinear fractional differential equation:

$$N[u(x,t)] = 0,$$  \hspace{1cm} (7)

where $N$ is a fractional differential operator $x$ and $t$ denotes independent variables, $u(x,t)$ is an unknown function. For simplicity, we ignore all boundary or initial condition, which can be treated in the same way.

By means of generalizing the topological concept of homotopy, (Liao (1992, 1999, 1999b)) constructed the zero-order deformation equation as

$$(1 - q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar H(x,t)N(\phi(x,t;q)),$$  \hspace{1cm} (8)

where $\hbar \neq 0$ denotes an auxiliary parameter, $H(x,t)$ is auxiliary function, $q \in [0,1]$ is an embedding parameter, $L$ is an auxiliary linear integer-order
operator and it possesses the property $L(C) = 0$, $u_0(x,t)$ is an initial guess of $u(x,t)$, $\varphi(x,t;q)$ is a kind of mapping of $u(x,t)$. It is important that one has great freedom to choose auxiliary parameter $h$ in homotopy analysis method. If $q = 0$ and $q = 1$, it holds

$$\varphi(x,t;0) = u_0(x,t) = u(x,0), \quad \varphi(x,t;1) = u(x,t),$$

(9)

Thus as $q$ increases from 0 to 1, the solution $\varphi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\varphi(x,t;q)$ in Taylors series with respect to $q$, one has

$$\varphi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} q^m u_m(x,t),$$

(10)

where

$$u_m(x,t) = \frac{\partial^m \varphi(x,t;q)}{\partial q^m} \bigg|_{q=0}$$

(11)

If the auxiliary linear integer order operator, the initial guess, and the auxiliary parameter $h$ are so properly chosen, the series (10) converges at $q = 1$, one has

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t),$$

(12)
According to the equation (1.4.5), the governing equation can be deduced from the zero-order deformation, Eq. (1.4.2). Define the vectors

\[ \vec{u}_n(x,t) = [u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)] \]  (13)

Differentiating Eq. (13) \( m \) times with respect to the embedding parameter \( q \) and then setting \( q = 0 \) and the finally dividing them by \( m! \), we have the so-called \( m \text{-th} \)–order deformation equation

\[ L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t)R_m[\vec{u}_{m-1}(x,t)], \]  (14)

where

\[ R_m[\vec{u}_{m-1}(x,t)] = \left. \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\Phi(x,t; q)]}{\partial q^{m-1}} \right|_{q=0} \]  (15)

and

\[ \chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1. \end{cases} \]  (16)

The \( m \text{-th} \)-order deformation equation (14) is linear and thus can be easily solved.