Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders (including complex orders), and their applications in science, engineering, mathematics, economics, and other fields. Since then many great mathematicians (pure and applied) of their times, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann M. Riesz, and H. Weyl, have contributed to this field. The history of Fractional Calculus (FC) goes back more than three centuries, when in 1695 the derivative of order $\alpha = 1/2$ described by Leibniz notes in his list of L’ Hospital dated 30 September 1695. Leibniz’s note led to the appearance of the theory of derivatives and integrals of arbitrary order. Since then, the new theory turned out to be very attractive to mathematicians and many different forms of fractional operators were introduced: the Grunwald–Letnikov, Riemann–Liouville, Riesz, and Caputo fractional derivatives. For three centuries the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics useful only for mathematicians. However, in the last few decades many authors pointed out that derivatives and integrals of non-integer are very
suitable for the description of properties of various real materials, e.g. polymers. Fractional calculus found many applications in various fields of physical sciences such as viscoelasticity, diffusion, control, relaxation processes and so on (Oldham and Spanier 1974; Miller and Ross 1993; Samko et al. 1993; Hilfer 2000; Podlubny 1999).

Nonlinear phenomena have important effects on applied mathematics, physics and issues related to engineering; many such physical phenomena are modeled in terms of nonlinear partial differential equations. Differential equations, integral equations or combinations of them, integro-differential equations, are obtained in modeling of real-life engineering phenomena that are inherently nonlinear with variable coefficients. Most of these types of equations do not have an analytical solution. Thus seeking solutions of nonlinear ordinary and partial differential equations of arbitrary order is still a significant problem that needs new techniques to develop exact and approximate solutions.

The main aim of this synopsis, we discuss standard analytical approaches (such as homotopy analysis method, homotopy analysis transform method, homotopy perturbation method, homotopy perturbation transform method and residual power series method) to the problem of fractional
derivatives and fractional integrals (simply called differ integrals), namely the Riemann-Liouville, the Caputo and the sequential approaches. The homotopy analysis method, introduced and applied first by Liao (1992, 1997, 2003, 2004a, 2004b, 2009, 2010a, 2010b), is a general approximate analytic approach used to obtain series solutions of nonlinear equations of various types, including algebraic equations, ordinary differential equations, partial differential equations, differential integral equations, differential difference equations, and coupled such equations. This method is valid no matter whether a nonlinear problem contains small/large physical parameters or not, which is essentially required in perturbation techniques. More importantly, unlike all perturbation and traditional non-perturbation methods, the homotopy analysis method provides us with both the freedom to choose proper base functions for approximating a nonlinear problem and a simple way to ensure the convergence of the solution series.

After the publication of Liao's book (1992) on the homotopy analysis method, a number of researchers have successfully applied this method to various nonlinear problems in science and engineering; for an extensive monograph one can refer to Liao (1992, 2009). As described therein, briefly speaking, by means of the homotopy analysis approach, one constructs a
continuous mapping of an initial guessed approximation to the exact solution of the equations considered. An auxiliary linear operator is chosen for constructing such a continuous mapping, and an auxiliary parameter is used to ensure the convergence of the solution series. The method enjoys great freedom in choosing initial approximations and auxiliary linear operators. By means of this kind of freedom, a complicated nonlinear problem can be transformed into an infinite number of simpler, linear sub-problems, which is the advantage of the method in this computer age.