# DOMINATION IN HYPERGRAPHS

## SYNOPSIS

This thesis embodies the work done by Bibin K. Jose under the guidance of Professor S. Arumugam

The thesis consists of four chapters.

1. Preliminaries
2. Domination Chain in Hypergraphs
3. Equality of Domination and Transversal Numbers in Hypergraphs
4. Hypergraph Domination and Strong Independence

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [8].

A vertex $u$ is called a *neighbor* of a vertex $v$ in $G$, if $uv$ is an edge of $G$. The set of all neighbors of $v$ is the *open neighborhood* of $v$ and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of $v$ in $G$. If $S \subseteq V$, then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. A vertex of degree one is a *pendant vertex* or a *leaf*. An edge $e$ in a graph $G$ is called a *pendant edge* if it is incident with a pendant vertex. Any vertex which is adjacent to a pendant vertex is called a *support vertex* or a *stem*. One of the major research areas in graph theory is the study of domination and related subset problems such as independence, irredundance, covering and matching. An excellent treatment of fundamentals of domination is given in the book by Haynes et al. [10]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [11].

A subset $S$ of $V$ is called an *independent set* of $G$ if no two vertices of $S$ are adjacent in $G$. A subset $S$ of $V$ is called a *dominating set* of $G$ if every vertex in $V \backslash S$ is adjacent to at least one vertex in $S$. If further, $S$ is an independent set then $S$ is called an *independent dominating set* of $G$. A dominating set $S$ is a minimal dominating set if no proper subset of $S$ is a dominating set. The minimum cardinality of a dominating set is called the *domination number* of $G$ and is denoted by $\gamma(G)$. The maximum cardinality of a minimal dominating set in $G$ is called the *upper domination number* of $G$ and is denoted by $\Gamma(G)$. The minimum cardinality of an independent dominating set is called the *independent domination number* of $G$ and is denoted by $i(G)$. The maximum cardinality of an independent set in $G$ is called the *independence number* of $G$ and is denoted by $\beta_0(G)$. Let $S$ be a set of vertices of a graph $G$ and let $u \in S$. We say that a vertex $v$ is a *private neighbor* of $u$ (with respect to $S$) if $N[v] \cap S = \{u\}$. The *private neighbor set* of $u$ with respect to $S$ is defined as $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. A set $S$ of vertices is *irredundant* if for every vertex $v \in S$, $pn[v, S] \neq \emptyset$. An irredundant set $S$ is called a *maximal irredundant* set if no proper superset of $S$ is irredundant. The minimum cardinality of a maximal irredundant set in a graph $G$ is called the *irredundance number* of $G$ and is denoted by $ir(G)$. The maximum cardinality of an irredundant set in a graph $G$ is called the *upper irredundance number* of $G$ and is denoted by $IR(G)$. The six parameters of domination, independence and irredundance are connected by a chain of inequalities as for any graph $G$, $ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G)$.

This inequality chain is called the *domination chain* of $G$ and was first observed by Cockayne et al. [4] since then this inequality chain has become one of the strongest focal points of research in domination.

A discrete structure which is a natural generalization of graphs is hypergraphs. For basic terminology in hypergraphs we refer to Berge [7].

Given a set $X$, a *hypergraph* $\mathcal{H}$ is a pair $\mathcal{H} = (X, \mathcal{E})$ where $\mathcal{E}$ is a collection of subsets of $X$. The elements of $X$ and of $\mathcal{E}$ are called *vertices* and *edges*, respectively. Traditionally, $E \neq \emptyset$ is required for all $E \in \mathcal{E}$, and we shall also assume that $X$ itself is not an edge; that is, $1 \leq |E| \leq |X| - 1$ for every edge $E$. In this thesis we consider only hypergraphs with finite number of vertices.

Given a hypergraph $\mathcal{H} = (X, \mathcal{E})$ and a nonempty subset $S \subseteq X$, the hypergraph $\mathcal{H}[S] = (S^-, \{E \in \mathcal{E} : E \subseteq S\})$ is called the *subhypergraph induced by* $S$ in $\mathcal{H}$, where $S^- \subseteq S$ is the set of vertices contained in at least one edge $E \subseteq S$. We say that $\mathcal{H}$ is of *rank* $k$ if $|E| \leq k$ holds for each edge. For $x \in X$, the *star* $\mathcal{H}(x)$ with center $x$ is a partial hypergraph formed by the edges containing $x$. The *vertex degree* $d(x)$ of $x$ is the number of vertices of $\mathcal{H}(x)$. Two vertices $v$ and $w$ are *adjacent* if there exists an edge $E \in \mathcal{E}$ that contains both $v$ and $w$, and non-adjacent otherwise. Given any vertex $v$ in a hypergraph $\mathcal{H} = (X, \mathcal{E})$, the set $N[v] = \{u \in X : u$ is adjacent to $v\} \cup \{v\}$ is called the *closed neighborhood* of $v$ in $\mathcal{H}$ and each vertex in the set is called a *neighbor* of $v$. The *open neighborhood* of the vertex $v$ is the set $N[v] - \{v\}$. A hypergraph is called *linear* if any two of its edges intersect in at most one vertex. A hypergraph is $k$*-uniform* if $|E| = k$ for all $E \in \mathcal{E}$.

Given an integer $k > 0$, the *k-section* of $\mathcal{H}$ is defined as the hypergraph $\mathcal{H}_{(k)} = (X, \mathcal{E}_{(k)})$ with edge set

$$\mathcal{E}_{(k)} = \{F \mid F \subset X,\ 1 \leq |F| \leq k,\ F \subset E \ for\ some\ E \in \mathcal{E}\}.$$

So, the 2-section $\mathcal{H}_{(2)}$ of $\mathcal{H}$ is a graph with the same vertices as $\mathcal{H}$, and with a loop attached to each vertex. We denote by $[\mathcal{H}]_2$ the graph obtained from this 2-section by *omitting loops.*

A *cycle of length* $q$ in $\mathcal{H}$ is a sequence $(x_1, E_1, x_2, E_2, \ldots, x_q, E_q, x_{q+1})$ such that $q > 1$, $x_1, x_2, \ldots, x_q$ are all distinct vertices of $\mathcal{H}$, $E_1, E_2, \ldots, E_q$ are all distinct edges of $\mathcal{H}$, $x_k, x_{k+1} \in E_k$ for $k = 1, 2, \ldots, q$ and $x_{q+1} = x_1$.

If $q$ is odd, then the cycle is called an *odd cycle.* A hypergraph is said to be *balanced* if every odd cycle in $\mathcal{H}$ has an edge containing three vertices of the cycle.

The *complement* of $\mathcal{H}$ is defined as $\overline{\mathcal{H}} = (X, \overline{\mathcal{E}})$, where $\overline{\mathcal{E}} = \{X \setminus E \mid E \in \mathcal{E}\}$.

In this thesis whenever we consider $\overline{\mathcal{H}}$, we restrict our attention to hypergraphs satisfying the condition that every vertex of $\mathcal{H}$ is incident with some edge of cardinality at least 2 and $|X| \geq 4$, in order to avoid the need to discuss trivial anomalies.

A transversal of a hypergraph $\mathcal{H}$ is defined to be a subset $T$ of the vertex set such that $T \cap E \neq \emptyset$ for every edge $E$ of $\mathcal{H}$. The transversal number of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the minimum number of vertices in a transversal.

Though domination in graphs is extensively studied, the corresponding concept in hypergraphs has not received much attention. In fact the concept of domination in hypergraphs was introduced recently by Acharya [1].

A set $S \subseteq X$ of a hypergraph $\mathcal{H}$ is said to be *independent* if every pair of vertices in $S$ is nonadjacent in $\mathcal{H}$. A subset $D$ of $X$ is called a *dominating set* of $\mathcal{H}$ if every vertex in $X \backslash D$ is adjacent to at least one vertex in $D$. If further, $D$ is an independent set then $D$ is called an *independent dominating set* of $\mathcal{H}$. The minimum cardinality of a dominating set is called the *domination number* of $\mathcal{H}$ and is denoted by $\gamma(\mathcal{H})$.

In this thesis we present several results relating to domination in hypergraphs. In Chapter 1, we collect some basic definitions and theorems on graphs and hypergraphs, which are needed for the subsequent chapters.

In Chapter 2, we introduce the analogous concept of domination chain for hypergraphs.

A dominating set $D$ is a minimal dominating set if no proper subset of $D$ is a dominating set. The maximum cardinality of a minimal dominating set in $\mathcal{H}$ is called the *upper domination number* of $\mathcal{H}$ and is denoted by $\Gamma(\mathcal{H})$. The minimum cardinality of an independent dominating set is called the *independent domination number* of $\mathcal{H}$ and is denoted by $i(\mathcal{H})$. The maximum cardinality of an independent set in $\mathcal{H}$ is called the *independence number* of $\mathcal{H}$ and is denoted by $\beta_0(\mathcal{H})$.

Let $S$ be a set of vertices of a hypergraph $\mathcal{H}$ and let $u \in S$. We say that a vertex $v$ is a *private neighbor* of $u$ (with respect to $S$) if $N[v] \cap S = \{u\}$. The *private neighbor set* of $u$ with respect to $S$ is defined as $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$.

A set $S$ of vertices is *irredundant* if for every vertex $v \in S$, $pn[v, S] \neq \emptyset$. An irredundant set $S$ is called a *maximal irredundant* set if no proper superset of $S$ is irredundant.

The minimum cardinality of a maximal irredundant set in a hypergraph $\mathcal{H}$ is called the *irredundance number* of $\mathcal{H}$ and is denoted by $ir(\mathcal{H})$. The maximum cardinality of an irredundant set in a hypergraph $\mathcal{H}$ is called the *upper irredundance number* of $\mathcal{H}$ and is denoted by $IR(\mathcal{H})$.

The six parameters of domination, independence and irredundance are connected by the following chain of inequalities.

$$ir(\mathcal{H}) \leq \gamma(\mathcal{H}) \leq i(\mathcal{H}) \leq \beta_0(\mathcal{H}) \leq \Gamma(\mathcal{H}) \leq IR(\mathcal{H}).$$

Further we obtain lot of results regarding the domination chain in hypergraphs.

Some of our results are relatives of the Nordhaus–Gaddum theorem, concerning the sums and products of domination parameters in hypergraphs and their complements.

The main results in Chapter 2 are the following.

**Theorem 1.** *For any hypergraph $\mathcal{H}$ with each edge has at least $k$ vertices, $\gamma(\mathcal{H}) \leq \frac{|X|}{k}$.*

**Theorem 2.** *For any hypergraph $\mathcal{H}$ with each edge has at least $k$ vertices, $\gamma(\mathcal{H}) \leq \frac{n+km}{2k}$ where $n$ and $m$ are number of vertices and edges respectively.*

**Theorem 3.** *An independent set $S$ of a hypergraph $\mathcal{H}$ is maximal independent if and only if it is independent and dominating.*

**Theorem 4.** $\beta_0(\mathcal{H}) = \beta_0([\mathcal{H}]_2)$ *where $[\mathcal{H}]_2$ is the 2-section of $\mathcal{H}$.*

**Theorem 5.** *A dominating set $S$ of a hypergraph $\mathcal{H}$ is a minimal dominating set of the hypergraph $\mathcal{H}$ if and only if it is dominating and irredundant.*

**Theorem 6.** *For any hypergraph $\mathcal{H}$, $IR(\mathcal{H}) + \delta(\mathcal{H}) \leq |X|$, where $\delta(\mathcal{H})$ is the minimum vertex degree of $\mathcal{H}$.*

**Theorem 7.** *For a hypergraph* $\mathcal{H}$
*(i)* $2 \le \gamma(\mathcal{H}) + \gamma(\overline{\mathcal{H}}) \le max\{4, \lfloor \frac{n}{2} \rfloor + 1\}$.
*(ii)* $1 \le \gamma(\mathcal{H})\gamma(\overline{\mathcal{H}}) \le max\{4, \lfloor \frac{n}{2} \rfloor\}$ *and the bounds are attainable.*

The inequality $\tau(\mathcal{H}) \ge \gamma(\mathcal{H})$ is valid for every hypergraph $\mathcal{H}$. In Chapter 3 we investigate the structure of graphs and hypergraphs satisfying $\tau(\mathcal{H}) = \gamma(\mathcal{H})$.

In 1981, Laskar and Walikar [12] raised the problem of characterization of graphs satisfying $\tau = \gamma$. First, Hartnell and Rall answered the question in [9], but their characterization was quite complicated. Randerath and Volkmann established another characterization in [13], but it was precise only for graphs of minimum degree at least 2; this characterization was recently completed by Wu and Yu [14] for the case of $\delta = 1$.

In Chapter 3, we give the characterization in a unified simpler way, avoiding redundances. This is a similar but different formulation from those in [13] and [14]. We need the following notions.

A vertex $v$ of a graph $G = (V, E)$ is called *stem* if $v$ is adjacent to a vertex of degree 1. The set of all stems of $G$ is denoted by Stem($G$) [14]. Let $\mathcal{S}(G)$ denote the graph obtained from $G$ by deleting all edges contained entirely in Stem($G$). Note that transformation $\mathcal{S}$ does not create isolated vertices, unless $G$ contains some component isomorphic to $K_2$.

**Theorem 8.** *For a connected graph $G$ of order at least 3, $\tau(G) = \gamma(G)$ holds if and only if there exists a bipartition $(A, B)$ of $\mathcal{S}(G)$ such that, $Stem(G) \subseteq A$, moreover for every pair $u, v \in A \backslash Stem(G)$, if $u$ and $v$ have some common neighbor then they have at least two common neighbors of degree two.*

We also investigate the structure of hypergraphs satisfying $\tau(\mathcal{H}) = \gamma(\mathcal{H})$ and prove that the corresponding recognition problem is NP -hard already on the class of 3-uniform linear hypergraphs. Structurally we focus our attention on hypergraphs in which each subhypergraph $\mathcal{H}'$ without isolated vertices fulfills the equality $\tau(\mathcal{H}') = \gamma(\mathcal{H}')$. It is shown that if each induced subhypergraph satisfies the equality then it holds for the non-induced ones as well. Moreover, we prove that for every positive integer $k$, there are only a finite number of forbidden subhypergraphs of rank $k$, and each of them has domination number at most $k$.

We summarize the main results of Chapter 3.

**Theorem 9.** *For all integer $n \geq 1$, there exists a hypergraph $\mathcal{H}$ such that $\gamma(\mathcal{H}) = \tau(\mathcal{H}) = n$.*

**Theorem 10.** *For a hypergraph $\mathcal{H}$, the equality $\tau = \gamma$ holds hereditarily if and only if it holds induced-hereditarily.*

**Theorem 11.** *For every hypergraph $\mathcal{H}$ which is minimal for $\tau > \gamma$, the equality $\tau(\mathcal{H}) = \gamma(\mathcal{H}) + 1$ holds.*

**Proposition 12.** *Any hypergraph $\mathcal{H}$ of rank $k$ ($k \geq 2$) can be embedded as an induced subhypergraph of a hypergraph $\mathcal{H}^*$ of rank $k$ with $\gamma(\mathcal{H}^*) = \tau(\mathcal{H}^*)$. Thus, there is no forbidden subhypergraph characterization for hypergraphs satisfying $\tau = \gamma$.*

**Theorem 13.** *If $\mathcal{H}$ is a hypergraph of rank $k$ and it is induced-minimal for $\tau > \gamma$, then $\gamma(\mathcal{H}) \leq k$.*

**Theorem 14.** *For every $k$, there exist only finitely many hypergraphs of rank $k$ which are minimal for $\tau > \gamma$.*

**Theorem 15.** *For a graph $G$ the equality $\tau = \gamma$ hereditarily holds if and only if $G$ contains no subgraph isomorphic to the triangle $K_3$, the 5-cycle $C_5$, and the path $P_6$ on six vertices.*

**Theorem 16.** *It can be decided in time $O(\sum_{v\in V} d^2(v))$ whether a generic input graph without isolated vertices satisfies $\tau = \gamma$.*

In Chapter 4, we restrict our attention to hypergraphs with $|X| \geq 4$, satisfying the following conditions.

(1) Every vertex of $\mathcal{H}$ is incident with some edge of cardinality at least 2.

(2) For every vertex $x$ of $\mathcal{H}$ there is an edge of cardinality at most $|X| = 2$ avoiding $x$.

Acharya [2] defined the following notions.

Let $D^o(\mathcal{H})$ and $D^m(\mathcal{H})$ be the set of all minimum dominating sets (of cardinality $\gamma(\mathcal{H})$) and set of all (inclusion-wise) minimal dominating sets, respectively.

Let $D \in D^o(\mathcal{H})$. An *inverse dominating set* with respect to $D$ is any dominating set $D'$ of $\mathcal{H}$ such that $D' \subseteq X \setminus D$. The *inverse domination number* of $\mathcal{H}$ is defined as

$$\gamma^{-1}(\mathcal{H}) = \min\{|D'| \ \mid \ D \in D^o(\mathcal{H}),\ D' \text{ is an inverse dominating set with respect to } D\}$$

Furthermore,

$$\gamma\gamma(\mathcal{H}) = \min\{|S_1| + |S_2| \mid S_1, S_2 \in D^m(\mathcal{H}),\ S_1 \cap S_2 = \phi\}$$

is called the *disjoint domination number* of $\mathcal{H}$.

Acharya [2] raised the following open problems and Conjectures in his paper.

**Problem 17.** *Find attainable lower and upper bounds for $\gamma\gamma(\mathcal{H}) + \gamma\gamma(\overline{\mathcal{H}})$*

**Conjecture 18.** *For any hypergraph $\mathcal{H}$, if $\gamma(\mathcal{H}) = i(\mathcal{H})$ then $\gamma\gamma(\mathcal{H}) = \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$*

**Conjecture 19.** *Every balanced hypergraph has a pair of disjoint maximal independent sets*

**Problem 20.** *characterize hypergraphs having two disjoint maximal strongly independent sets*

In this chapter, we solve the Conjectures 18, 19 and open Problems 17, 20. Some of our results are relatives of the Nordhaus–Gaddum theorem, concerning the sum of domination parameters in hypergraphs and their complements.

We solve Problem 17 in Section 4.3, and for the upper bound we prove even a stronger statement. Namely, we prove that $\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) + \gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}}) \leq \max\{8, n+2\}$ holds, which is tight for all $n \geq 4$ also for $\gamma\gamma(\mathcal{H}) + \gamma\gamma(\overline{\mathcal{H}})$. Moreover, $\min\{\gamma\gamma(\mathcal{H}), \gamma\gamma(\overline{\mathcal{H}})\} = \min\{\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}), \gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}})\} \leq 4$.

On the other hand, we disprove Conjecture 18 by giving an infinite family of counterexamples. Also, we give counterexamples to Conjecture 19. Moreover, we observe that Problem 20 about the characterization of hypergraphs having two disjoint maximal strongly independent sets, is reducible to the same problem on graphs.

The main results in Chapter 4 are the following.

**Theorem 21.** *For every integer $n \geq 4$,*

$$4 \leq \gamma\gamma(\mathcal{H}) + \gamma\gamma(\overline{\mathcal{H}}) \leq \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) + \gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}}) \leq \max\{8, n+2\}$$

*and the bounds are tight.*

**Theorem 22.** *For every $n \geq 4$ and every hypergraph $\mathcal{H}$ of order $n$,*

$$\min\{\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}), \gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}})\} \leq 4$$

*and the bound is tight.*

**Proposition 23.** *For every integer $n \geq 7$, there exists a hypergraph $\mathcal{H}$ of order $n$ with $\gamma(\mathcal{H}) = i(\mathcal{H})$ but $\gamma\gamma(\mathcal{H}) \neq \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$.*

**Theorem 24.** *For every integer $k \geq 1$ there exists a connected hypergraph $\mathcal{H}$ such that $\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) - \gamma\gamma(\mathcal{H}) = k$.*

## References


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